REAL-VARIABLE CHARACTERIZATIONS OF BERGMAN SPACES

ZEQIAN CHEN AND WEI OUYANG

Abstract. In this paper, we give a survey of results obtained recently by the present authors on real-variable characterizations of Bergman spaces, which are closely related to maximal and area integral functions in terms of the Bergman metric. In particular, we give a new proof of those results concerning area integral characterizations through using the method of vector-valued Calderón-Zygmund operators to handle Bergman singular integral operators on the complex ball. The proofs involve some sharp estimates of the Bergman kernel function and Bergman metric.

1. Introduction

There is a mature and powerful real variable Hardy space theory which has distilled some of the essential oscillation and cancellation behavior of holomorphic functions and then found that behavior ubiquitous. A good introduction to that is [11]; a more recent and fuller account is in [1, 14, 16] and references therein. However, the real-variable theory of the Bergman space is less well developed, even in the case of the unit disc (cf. [12]).

Recently, in [7] the present authors established real-variable type maximal and area integral characterizations of Bergman spaces in the unit ball of $\mathbb{C}^n$. The characterizations are in terms of maximal functions and area functions on Bergman balls involving the radial derivative, the complex gradient, and the invariant gradient. Subsequently, in [8] we introduced a family of holomorphic spaces of tent type in the unit ball of $\mathbb{C}^n$ and showed that those spaces coincide with Bergman spaces. Moreover, the characterizations extend to cover Besov-Sobolev spaces. A special case of this is a characterization of H^p spaces involving only area functions on Bergman balls.

We remark that the first real-variable characterization of the Bergman spaces was presented by Coifman and Weiss in 1970's. Recall that

$$\varrho(z,w) = \begin{cases} \big||z|-|w|\big| + \Big|1 - \dfrac{1}{|z||w|}\langle z, w\rangle\Big|, & \text{if } z, w \in \mathbb{B}_n \backslash \{0\}, \\ |z|+|w|, & \text{otherwise} \end{cases}$$

2010 Mathematics Subject Classification: 46E15; 32A36, 32A50.

Key words: Bergman space, Bergman metric, Maximal function, Area function, Bergman integral operator.

is a pseudo-metric on $\mathbb{B}_n$ and $(\mathbb{B}_n, \varrho, dv_\alpha)$ is a homogeneous space. By their theory of harmonic analysis on homogeneous spaces, Coifman and Weiss [11] can use ϱ to obtain a real-variable atomic decomposition for Bergman spaces. However, since the Bergman metric β underlies the complex geometric structure of the unit ball of $\mathbb{C}^n$, one would prefer to real-variable characterizations of the Bergman spaces in terms of β. Clearly, the results obtained in [7, 8] are such a characterization.

In this paper, we will give a detailed survey of results obtained in [7, 8]. Moreover, we will give a new proof of those results concerning area integral characterizations through using the method of vector-valued Calderón-Zygmund operator theory to handle Bergman singular integral operators on the complex ball. This paper is organized as follows. In Section 2, some notations and a number of auxiliary (and mostly elementary) facts about the Bergman kernel functions are presented. In Section 3, we will discuss real-variable type atomic decomposition of Bergman spaces. In particular, we will present the atomic decomposition of Bergman spaces with respect to Carleson tubes that was obtained in [7]. Section 4 is devoted to present maximal and area integral function characterizations of Bergman spaces. Finally, in Section 5, we will give a new proof of those results concerning the area integral characterizations obtained in [7, 8] using the argument of Calderón-Zygmund operator theory through introducing Bergman singular integral operators on the complex ball.

In what follows, C always denotes a constant depending (possibly) on n, q, p, γ or α but not on f, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y, by $X \lesssim Y$ we mean that there exists a constant $C > 0$ such that $X \le CY$. We denote by $X \approx Y$ when $X \lesssim Y$ and $Y \lesssim X$. Any notation and terminology not otherwise explained, are as used in [20] for spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$.

2. Bergman spaces

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n, and let $\mathbb{B}_n$ denote the open unit ball in $\mathbb{C}^n$. The boundary of $\mathbb{B}_n$ will be denoted by $\mathbb{S}_n$ and is called the unit sphere in $\mathbb{C}^n$. Also, we denote by $\overline{\mathbb{B}}_n$ the closed unit ball, i.e., $\overline{\mathbb{B}}_n = \{z \in \mathbb{C}^n : \ |z| \le 1\} = \mathbb{B}_n \cup \mathbb{S}_n$. The automorphism group of $\mathbb{B}^n$, denoted by $\mathrm{Aut}(\mathbb{B}^n)$, consists of all bi-holomorphic mappings of $\mathbb{B}^n$. Traditionally, bi-holomorphic mappings are also called automorphisms.

For $\alpha \in \mathbb{R}$, the weighted Lebesgue measure dv_α on $\mathbb{B}_n$ is defined by

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z)$$

where $c_\alpha = 1$ for $\alpha \le -1$ and $c_\alpha = \Gamma(n + \alpha + 1)/[n!\Gamma(\alpha + 1)]$ if $\alpha > -1$, which is a normalizing constant so that dv_α is a probability measure on $\mathbb{B}_n$.

In the case of $\alpha = -(n+1)$ we denote the resulting measure by

$$d\tau(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}},$$

and call it the invariant measure on $\mathbb{B}^n$, since $d\tau = d\tau \circ \varphi$ for any automorphism φ of $\mathbb{B}^n$.

Recall that for $\alpha > -1$ and $p > 0$ the (weighted) Lebesgue space $L^p_\alpha(\mathbb{B}_n)$ (or, L^p_α in short) consists of measurable (complex) functions f on $\mathbb{B}_n$ with

$$\|f\|_{p,\,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z)\right)^{\frac{1}{p}} < \infty.$$

The (weighted) Bergman space $\mathcal{A}^p_\alpha$ is then defined as

$$\mathcal{A}^p_\alpha = \mathcal{H}(\mathbb{B}_n) \cap L^p_\alpha,$$

where $\mathcal{H}(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. When $\alpha = 0$ we simply write $\mathcal{A}^p$ for $\mathcal{A}^p_0$. These are the usual Bergman spaces. Note that for $1 \le p < \infty$, $\mathcal{A}^p_\alpha$ is a Banach space under the norm $\| \; \|_{p,\,\alpha}$. If $0 < p < 1$, the space $\mathcal{A}^p_\alpha$ is a quasi-Banach space with p-norm $\|f\|^p_{p,\alpha}$.

Recall that the dual space of $\mathcal{A}^1_\alpha$ is the Bloch space $\mathcal{B}$ defined as follows (we refer to [20] for details). The Bloch space $\mathcal{B}$ of $\mathbb{B}_n$ is defined to be the space of holomorphic functions f in $\mathbb{B}_n$ such that

$$\|f\|_{\mathcal{B}} = \sup\{\, |\widetilde{\nabla} f(z)| : z \in \mathbb{B}_n\} < \infty.$$

$\| \; \|_{\mathcal{B}}$ is a semi-norm on $\mathcal{B}$. $\mathcal{B}$ becomes a Banach space with the following norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

It is known that the Banach dual of $\mathcal{A}^1_\alpha$ can be identified with $\mathcal{B}$ (with equivalent norms) under the integral pairing

$$\langle f, g\rangle_\alpha = \lim_{r\to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(z)} dv_\alpha(z), \quad f \in \mathcal{A}^1_\alpha,\; g \in \mathcal{B}.$$

(e.g., see Theorem 3.17 in [20].)

We define the so-called generalized Bergman spaces as follows (e.g., [19]). For $0 < p < \infty$ and $-\infty < \alpha < \infty$ we fix a nonnegative integer k with $pk + \alpha > -1$ and define $\mathcal{A}^p_\alpha$ as the space of all $f \in \mathcal{H}(\mathbb{B}_n)$ such that $(1 - |z|^2)^k \mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_\alpha)$. One then easily observes that $\mathcal{A}^p_\alpha$ is independent of the choice of k and consistent with the traditional definition when $\alpha > -1$. Let N be the smallest nonnegative integer such that $pN + \alpha > -1$ and define

$$\|f\|_{p,\alpha} = |f(0)| + \left(\int_{\mathbb{B}_n} (1-|z|^2)^{pN} |\mathcal{R}^N f(z)|^p dv_\alpha(z)\right)^{\frac{1}{p}}, \quad f \in \mathcal{A}^p_\alpha\,. \tag{2.1}$$

Equipped with (2.1), $\mathcal{A}^p_\alpha$ becomes a Banach space when $p \ge 1$ and a quasi-Banach space for $0 < p < 1$.

Note that the family of the generalized Bergman spaces $\mathcal{A}^p_\alpha$ covers most of the spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, which has been

extensively studied before in the literature under different names. For example, $B_p^s = \mathcal{A}_\alpha^p$ with $\alpha = -(ps+1)$, where B_p^s is the classical diagonal Besov space consisting of holomorphic functions f in $\mathbb{B}_n$ such that $(1-|z|^2)^{k-s}\mathcal{R}^k f$ belongs to $L^p(\mathbb{B}_n, dv_{-1})$ with k being any positive integer greater than s. It is clear that $\mathcal{A}_\alpha^p = B_p^s$ with $s = -(\alpha+1)/p$. Thus the generalized Bergman spaces $\mathcal{A}_\alpha^p$ are exactly the diagonal Besov spaces. On the other hand, if k is a positive integer, p is positive, and β is real, then there is the Sobolev space $W_{k,\beta}^p$ consisting of holomorphic functions f in $\mathbb{B}_n$ such that the partial derivatives of f of order up to k all belong to $L^p(\mathbb{B}_n, dv_\beta)$ (cf. [2, 3, 6]). It is easy to see that these holomorphic Sobolev spaces are in the scale of the generalized Bergman spaces, i.e., $W_{k,\beta}^p = \mathcal{A}_\alpha^p$ with $\alpha = -(pk-\beta+1)$ (e.g., see [19] for an overview). We refer to Arcozzi-Rochberg-Sawyer [4, 5], Tchoundja [17] and Volberg-Wick [18] for some recent results on such Besov spaces and more references.

Recall that $D(z,\gamma)$ denotes the Bergman metric ball at z

$$D(z,\gamma) = \{w \in \mathbb{B}_n \ : \ \beta(z,w) < \gamma\}$$

with $\gamma > 0$, where β is the Bergman metric on $\mathbb{B}_n$. It is known that

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z,w \in \mathbb{B}_n,$$

whereafter φ_z is the bijective holomorphic mapping in $\mathbb{B}_n$, which satisfies $\varphi_z(0) = z$, $\varphi_z(z) = 0$ and $\varphi_z \circ \varphi_z = id$. If $\mathbb{B}_n$ is equipped with the Bergman metric β, then $\mathbb{B}_n$ is a separable metric space. We shall call $\mathbb{B}_n$ a separable metric space instead of $(\mathbb{B}_n, \beta)$.

For reader's convenience we collect some elementary facts on the Bergman metric and holomorphic functions in the unit ball of $\mathbb{C}^n$.

Lemma 2.1. *(cf. Lemma 1.24 in* [20]*) For any real α and positive γ there exist constant C_γ such that*

$$C_\gamma^{-1}(1-|z|^2)^{n+1+\alpha} \le v_\alpha(D(z,\gamma)) \le C_\gamma(1-|z|^2)^{n+1+\alpha}$$

for all $z \in \mathbb{B}_n$.

Lemma 2.2. *(cf. Lemma 2.20 in* [20]*) For each $\gamma > 0$,*

$$1-|a|^2 \approx 1-|z|^2 \approx |1-\langle a,z\rangle|$$

for all a in $\mathbb{B}_n$ with $z \in D(a,\gamma)$.

Lemma 2.3. *(cf. Lemma 2.27 in* [20]*) For each $\gamma > 0$,*

$$|1-\langle z,u\rangle| \approx |1-\langle z,v\rangle|$$

for all z in $\bar{\mathbb{B}}_n$ and u,v in $\mathbb{B}_n$ with $\beta(u,v) < \gamma$.

3. Atomic decomposition

We first recall the following "complex-variable" atomic decomposition for Bergman spaces due to Coifman and Rochberg [10] (see also [20], Theorem 2.30).

Theorem 3.1. *Suppose $p > 0, \alpha > -1$, and $b > n \max\{1, 1/p\} + (\alpha + 1)/p$. Then there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that $\mathcal{A}^p_\alpha$ consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{(1 - \langle z, a_k\rangle)^b}, \quad z \in \mathbb{B}_n, \tag{3.1}$$

where $\{c_k\}$ belongs to the sequence space ℓ^p and the series converges in the norm topology of $\mathcal{A}^p_\alpha$. Moreover,

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \approx \inf\Big\{\sum_k |c_k|^p\Big\},$$

where the infimum runs over all the above decompositions.

By Theorem 3.1 we conclude that for any $\alpha > -1$, $\mathcal{A}^p_\alpha$ as a Banach space is isomorphic to ℓ^p for every $1 \le p < \infty$.

Now we turn to the real-variable atomic decomposition of Bergman spaces. To this end, we need some more notations as follows.

For any $\zeta \in \mathbb{S}_n$ and $r > 0$, the set

$$Q_r(\zeta) = \{z \in \mathbb{B}_n : d(z, \zeta) < r\}$$

is called a Carleson tube with respect to the nonisotropic metric d. We usually write $Q = Q_r(\zeta)$ in short.

As usual, we define the atoms with respect to the Carleson tube as follows: for $1 < q < \infty$, $a \in L^q(\mathbb{B}_n, dv_\alpha)$ is said to be a $(1, q)_\alpha$-atom if there is a Carleson tube Q such that

(1) a is supported in Q;
(2) $\|a\|_{L^q(\mathbb{B}_n, dv_\alpha)} \le v_\alpha(Q)^{\frac{1}{q}-1}$;
(3) $\int_{\mathbb{B}_n} a(z)\, dv_\alpha(z) = 0$.

The constant function 1 is also considered to be a $(1, q)_\alpha$-atom.

Note that for any $(1, q)_\alpha$-atom a,

$$\|a\|_{1,\alpha} = \int_Q |a| dv_\alpha \le v_\alpha(Q)^{1-1/q} \|a\|_{q,\alpha} \le 1.$$

Then, we define $\mathcal{A}^{1,q}_\alpha$ as the space of all $f \in \mathcal{A}^1_\alpha$ which admits a decomposition

$$f = \sum_i \lambda_i P_\alpha a_i \quad \text{and} \quad \sum_i |\lambda_i| \le C_q \|f\|_{1,\alpha},$$

where for each i, a_i is an $(1,q)_\alpha$-atom and $\lambda_i \in \mathbb{C}$ so that $\sum_i |\lambda_i| < \infty$. We equip this space with the norm

$$\|f\|_{\mathcal{A}^{1,q}_\alpha} = \inf\Big\{ \sum_i |\lambda_i| : \; f = \sum_i \lambda_i P_\alpha a_i \Big\}$$

where the infimum is taken over all decompositions of f described above.

It is easy to see that $\mathcal{A}^{1,q}_\alpha$ is a Banach space.

Theorem 3.2. *Let $1 < q < \infty$ and $\alpha > -1$. For every $f \in \mathcal{A}^1_\alpha$ there exist a sequence $\{a_i\}$ of $(1,q)_\alpha$-atoms and a sequence $\{\lambda_i\}$ of complex numbers such that*

$$f = \sum_i \lambda_i P_\alpha a_i \quad \textit{and} \quad \sum_i |\lambda_i| \le C_q \|f\|_{1,\alpha}. \tag{3.2}$$

Moreover,

$$\|f\|_{1,\alpha} \approx \inf \sum_i |\lambda_i|$$

where the infimum is taken over all decompositions of f described above and "$\approx$" depends only on α and q.

Theorem 3.2 is proved in [7] via duality.

Remark 3.1. *One would like to expect that when $0 < p < 1$, $\mathcal{A}^p_\alpha$ also admits an atomic decomposition in terms of atoms with respect to Carleson tubes. However, the proof of Theorem 3.2 via duality cannot be extended to the case $0 < p < 1$. At the time of this writing, this problem is entirely open.*

As mentioned in Introduction, by their theory of harmonic analysis on homogeneous spaces, Coifman and Weiss [11] *have obtained a real-variable atomic decomposition in terms of ϱ for Bergman spaces in the case $0 < p \le 1$.*

4. Real-variable characterizations

4.1. **Maximal functions.** As is well known, maximal functions play a crucial role in the real-variable theory of Hardy spaces (cf. [16]). In [7], the authors established a maximal-function characterization for the Bergman spaces. To this end, we define for each $\gamma > 0$ and $f \in \mathcal{H}(\mathbb{B}_n)$:

$$(\mathrm{M}_\gamma f)(z) = \sup_{w \in D(z,\gamma)} |f(w)|, \; \forall z \in \mathbb{B}_n. \tag{4.1}$$

The following result is proved in [7].

Theorem 4.1. *Suppose $\gamma > 0$ and $\alpha > -1$. Let $0 < p < \infty$. Then for any $f \in \mathcal{H}(\mathbb{B}_n)$, $f \in \mathcal{A}^p_\alpha$ if and only if $\mathrm{M}_\gamma f \in L^p(\mathbb{B}_n, dv_\alpha)$. Moreover,*

$$\|f\|_{p,\alpha} \approx \|\mathrm{M}_\gamma f\|_{p,\alpha}, \tag{4.2}$$

where "$\approx$" depends only on γ, α, p, and n.

The norm appearing on the right-hand side of (4.2) can be viewed an analogue of the so-called nontangential maximal function in Hardy spaces. The proof of Theorem 4.1 is fairly elementary, using some basic facts and estimates on the Bergman balls.

Corollary 4.1. *Suppose $\gamma > 0$ and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and k be a nonnegative integer such that $pk + \alpha > -1$. Then for any $f \in \mathcal{H}(\mathbb{B}_n)$, $f \in \mathcal{A}^p_\alpha$ if and only if $\mathrm{M}_\gamma(\mathcal{R}^k f) \in L^p(\mathbb{B}_n, dv_\alpha)$, where*

$$\mathrm{M}_\gamma(\mathcal{R}^k f)(z) = \sup_{w \in D(z,\gamma)} |(1-|w|^2)^k \mathcal{R}^k f(w)|, \quad z \in \mathbb{B}_n. \tag{4.3}$$

Moreover,

$$\|f\|_{p,\alpha} \approx |f(0)| + \|\mathrm{M}_\gamma(\mathcal{R}^k f)\|_{p,\alpha}, \tag{4.4}$$

where "$\approx$" depends only on γ, α, p, k, and n.

To prove Corollary 4.1, one merely notices that $f \in \mathcal{A}^p_\alpha$ if and only if $\mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_{\alpha+pk})$ and applies Theorem 4.1 to $\mathcal{R}^k f$ with the help of Lemma 2.2.

4.2. **Area integral functions.** In order to state the real-variable area integral characterizations of the Bergman spaces, we require some more notation. For any $f \in \mathcal{H}(\mathbb{B}_n)$ and $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$ we define

$$\mathcal{R}f(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k}$$

and call it the radial derivative of f at z. The complex and invariant gradients of f at z are respectively defined as

$$\nabla f(z) = \Big(\frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_n}\Big) \text{ and } \widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0).$$

Now, for fixed $\gamma > 0$ and $1 < q < \infty$, we define for each $f \in \mathcal{H}(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$:

(1) The radial area function

$$A^{(q)}_\gamma(\mathcal{R}f)(z) = \left(\int_{D(z,\gamma)} |(1-|w|^2)\mathcal{R}f(w)|^q d\tau(w)\right)^{\frac{1}{q}}.$$

(2) The complex gradient area function

$$A^{(q)}_\gamma(\nabla f)(z) = \left(\int_{D(z,\gamma)} |(1-|w|^2)\nabla f(w)|^q d\tau(w)\right)^{\frac{1}{q}}.$$

(3) The invariant gradient area function

$$A^{(q)}_\gamma(\tilde{\nabla} f)(z) = \left(\int_{D(z,\gamma)} |\tilde{\nabla} f(w)|^q d\tau(w)\right)^{\frac{1}{q}}.$$

The following theorem is proved in [7].

Theorem 4.2. *Suppose $\gamma > 0, 1 < q < \infty$, and $\alpha > -1$. Let $0 < p < \infty$. Then, for any $f \in \mathcal{H}(\mathbb{B}_n)$ the following conditions are equivalent:*
(a) $f \in \mathcal{A}^p_\alpha$.
(b) $A^{(q)}_\gamma(\mathcal{R}f)$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.
(c) $A^{(q)}_\gamma(\nabla f)$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.
(d) $A^{(q)}_\gamma(\tilde{\nabla} f)$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.
Moreover, the quantities

$$|f(0)| + \|A^{(q)}_\gamma(\mathcal{R}f)\|_{p,\alpha},\ |f(0)| + \|A^{(q)}_\gamma(\nabla f)\|_{p,\alpha},\ |f(0)| + \|A^{(q)}_\gamma(\tilde{\nabla} f)\|_{p,\alpha},$$

are all comparable to $\|f\|_{p,\alpha}$, where the comparable constants depend only on $\gamma, q, \alpha, p,$ and n.

In particular, taking the equivalence of (a) and (b), one obtains

$$\|f\|_{p,\alpha} \approx |f(0)| + \|A^{(q)}_\gamma(\mathcal{R}f)\|_{p,\alpha},$$

which looks tantalizingly simple. However, the authors know no simple proof of this fact even in the case of the usual Bergman space on the unit disc.

Corollary 4.2. *Suppose $\gamma > 0, 1 < q < \infty$, and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and k be a nonnegative integer such that $pk + \alpha > -1$. Then for any $f \in \mathcal{H}(\mathbb{B}_n)$, $f \in \mathcal{A}^p_\alpha$ if and only if $A^{(q)}_\gamma(\mathcal{R}^{k+1} f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$, where*

$$A^{(q)}_\gamma(\mathcal{R}^k f)(z) = \left(\int_{D(z,\gamma)} \left|(1 - |w|^2)^k \mathcal{R}^k f(w)\right|^q d\tau(w)\right)^{\frac{1}{q}}. \tag{4.5}$$

Moreover,

$$\|f\|_{p,\alpha} \approx |f(0)| + \|A^{(q)}_\gamma(\mathcal{R}^{k+1} f)\|_{p,\alpha}, \tag{4.6}$$

where "$\approx$" depends only on $\gamma, q, \alpha, p, k,$ and n.

To prove Corollary 4.2, one merely notices that $f \in \mathcal{A}^p_\alpha$ if and only if $\mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_{\alpha+pk})$ and applies Theorem 4.2 to $\mathcal{R}^k f$ with the help of Lemma 2.2.

4.3. **Tent spaces.** The basic functional used below is the one mapping functions in $\mathbb{B}_n$ to functions in $\mathbb{B}_n$, given by

$$A^{(q)}_\gamma(f)(z) = \left(\int_{D(z,\gamma)} |f(w)|^q d\tau(w)\right)^{\frac{1}{q}} \tag{4.7}$$

if $1 < q < \infty$, and

$$A^{(\infty)}_\gamma(f)(z) = \sup_{w \in D(z,\gamma)} |f(w)|, \quad \text{when } q = \infty. \tag{4.8}$$

Then, the "holomorphic space of tent type" $T^p_{q,\alpha}$ in $\mathbb{B}_n$ is defined as the holomorphic functions f in $\mathbb{B}_n$ so that $A^{(q)}_\gamma(f) \in L^p_\alpha$, when $0 < p \leq \infty$ and $\alpha > -1,\ \gamma > 0, 1 < q \leq \infty$. The corresponding classes are then equipped

with a norm (or, quasi-norm) $\|f\|_{T^p_{q,\alpha}} = \|A^{(q)}_\gamma(f)\|_{p,\alpha}$. This motivation aries form the tent spaces in $\mathbb{R}^n$, which were introduced and developed by Coifman, Meyer and Stein in [9].

The case of $q=\infty$ and $0<p<\infty$ was studied in Section 4.1 (see [7] for details). Actually, the resulting tent type spaces $T^p_{\infty,\alpha}$ is Bergman spaces $\mathcal{A}^p_\alpha$. It is clear that $T^\infty_{q,\alpha}$ with $1<q<\infty$ is imbedded in Bloch space. On the other hand, $T^p_{q,\alpha}$ are Banach spaces when $p\geq 1$.

It is well known that the Hardy-Littlewood maximal function operator has played important role in harmonic analysis. To cater our estimates, we use two variants of the non-central Hardy-Littlewood maximal function operator acting on the weighted Lebesgue spaces $L^p_\alpha(\mathbb{B}_n)$, namely,

$$M^{(q)}_\gamma(f)(z) = \sup_{z\in D(w,\gamma)} \left(\frac{1}{v_\alpha(D(w,\gamma))}\int_{D(w,\gamma)} |f|^q dv_\alpha\right)^{\frac{1}{q}} \tag{4.9}$$

for $0<q<\infty$. We simply write $M_\gamma(f)(z) := M^{(1)}_\gamma(f)(z)$.

The following result is proved in [8].

Theorem 4.3. *Suppose $\gamma>0, 1<q<\infty$, and $\alpha>-1$. Let $0<p<\infty$. Then for any $f\in\mathcal{H}(\mathbb{B}_n)$, the following conditions are equivalent:*
(1) $f\in\mathcal{A}^p_\alpha$.
(2) $A^{(q)}_\gamma(f)$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.
(3) $M^{(q)}_\gamma(f)$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.
Moreover,

$$\|f\|_{\mathcal{A}^p_\alpha} \approx \|f\|_{T^p_{q,\alpha}} \approx \|M^{(q)}_\gamma(f)\|_{p,\alpha},$$

where the comparable constants depend only on γ, q, α, p, and n.

Note that the Bergman metric β is non-doubling on $\mathbb{B}^n$ and so $(\mathbb{B}_n, \beta, dv_\alpha)$ is a non-homogeneous space. The proof of the above theorem does involve some techniques of non-homogeneous harmonic analysis developed in [15].

Corollary 4.3. *Suppose $\gamma>0, 1<q<\infty$, and $\alpha\in\mathbb{R}$. Let $0<p<\infty$ and k be a nonnegative integer such that $pk+\alpha>-1$. Then for any $f\in\mathcal{H}(\mathbb{B}_n)$, $f\in\mathcal{A}^p_\alpha$ if and only if $A^{(q)}_\gamma(\mathcal{R}^k f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$ if and only if $M^{(q)}_\gamma(\mathcal{R}^k f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$, where*

$$A^{(q)}_\gamma(\mathcal{R}^k f)(z) = \left(\int_{D(z,\gamma)} \left|(1-|w|^2)^k \mathcal{R}^k f(w)\right|^q d\tau(w)\right)^{\frac{1}{q}} \tag{4.10}$$

and
(4.11)

$$M^{(q)}_\gamma(\mathcal{R}^k f)(z) = \sup_{z\in D(w,\gamma)} \left(\int_{D(w,\gamma)} \left|(1-|u|^2)^k \mathcal{R}^k f(u)\right|^q \frac{dv_\alpha(u)}{v_\alpha(D(w,\gamma))}\right)^{\frac{1}{q}}$$

Moreover,

$$\|f\|_{p,\alpha} \approx |f(0)| + \|A_\gamma^{(q)}(\mathcal{R}^k f)\|_{p,\alpha} \approx |f(0)| + \|M_\gamma^{(q)}(\mathcal{R}^k f)\|_{p,\alpha}, \tag{4.12}$$

where "$\approx$" depends only on $\gamma, q, \alpha, p, k,$ and n.

To prove Corollary 4.3, one merely notices that $f \in \mathcal{A}_\alpha^p$ if and only if $\mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_{\alpha+pk})$ and applies Theorem 4.3 to $\mathcal{R}^k f$ with the help of Lemma 2.2. When $\alpha > -1$, we can take $k = 1$ and then recover Theorem 4.2.

As mentioned in Section 2, the family of the generalized Bergman spaces $\mathcal{A}_\alpha^p$ covers most of the spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, such as the classical diagonal Besov space B_p^s and the Sobolev space $W_{k,\beta}^p$. In particular, $\mathcal{H}_s^p = \mathcal{A}_\alpha^p$ with $\alpha = -2s - 1$, where $\mathcal{H}_s^p$ is the Hardy-Sobolev space defined as the set

$$\left\{ f \in \mathcal{H}(\mathbb{B}_n) :\ \|f\|_{\mathcal{H}_s^p}^p = \sup_{0<r<1} \int_{\mathbb{S}_n} |(I+\mathcal{R})^s f(r\zeta)|^p d\sigma(\zeta) < \infty \right\}.$$

Here,

$$(I+\mathcal{R})^s f = \sum_{k=0}^{\infty} (1+k)^s f_k$$

if $f = \sum_{k=0}^\infty f_k$ is the homogeneous expansion of f. There are several real-variable characterizations of the Hardy-Sobolev spaces obtained by Ahern and Bruna [1] (see also [3]). These characterizations are in terms of maximal and area functions on the admissible approach region

$$D_\alpha(\eta) = \left\{ z \in \mathbb{B}_n :\ |1 - \langle z, \eta\rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}, \quad \eta \in \mathbb{S}_n,\ \alpha > 1.$$

Evidently, Corollary 4.3 present new real-variable descriptions of the Hardy-Sobolev spaces in terms of the Bergman metric. A special case of this is a characterization of the usual Hardy space $\mathcal{H}^p = \mathcal{A}_{-1}^p$ itself.

5. Bergman integral operators

5.1. **Vector-valued kernels and Calderón-Zygmund operators on homogeneous spaces.** Recall that a quasimetric on a set X is a map ρ from $X \times X$ to $[0, \infty)$ such that

(1) $\rho(x, y) = 0$ if and only if $x = y$;
(2) $\rho(x, y) = \rho(y, x)$;
(3) there exists a positive constant $C \geq 1$ such that

$$\rho(x, y) \leq C[\rho(x, z) + \rho(z, y)], \quad \forall x, y, z \in X,$$

(the quasi-triangular inequality).

For any $x \in X$ and $r > 0$, the set $B(x, r) = \{y \in X :\ \rho(x, y) < r\}$ is called a ρ-ball of center x and radius r.

A space of homogeneous type is a topological space X endowed with a quasimetrc ρ and a Borel measure μ such that

(a) for each $x \in X$, the balls $B(x,r)$ form a basis of open neighborhoods of x and, also, $\mu(B(x,r)) > 0$ whenever $r > 0$;
(b) (doubling property) there exists a constant $C > 0$ such that for each $x \in X$ and $r > 0$, one has
$$\mu(B(x,2r)) \leq C\mu(B(x,r)).$$

(X,ρ,μ) is called a space of homogeneous type or simply a homogeneous space. We will usually abusively call X a homogeneous space instead of (X,ρ,μ). We refer to [11, 16] for details on harmonic analysis on homogeneous spaces.

Let E be a Banach space. Let $L^p(\mu,E)$ be the usual Bochner-Lebesgue space for $1 \leq p \leq \infty$, and let $L^{1,\infty}(\mu,E)$ be defined by
$$L^{1,\infty}(\mu,E) := \big\{f:\ X \mapsto E | f \text{ is strongly measurable such that } \|f\|_{L^{1,\infty}(\mu,E)} < \infty\big\},$$
where $\|f\|_{L^{1,\infty}(\mu,E)} := \sup_{t>0} t\mu\big(\{x \in X : \|f(x)\|_E > t\}\big)$. Note that $\|f\|_{L^{1,\infty}}$ is not actually a norm in the sense that it does not satisfy the triangle inequality. However, we still have
$$\|cf\|_{L^{1,\infty}(\mu,E)} = |c|\|f\|_{L^{1,\infty}(\mu,E)} \text{ and } \|f+g\|_{L^{1,\infty}(\mu,E)} \leq 2(\|f\|_{L^{1,\infty}(\mu,E)} + \|g\|_{L^{1,\infty}(\mu,E)})$$
for every $c \in \mathbb{C}$ and $f,g \in L^{1,\infty}(\mu,E)$.

If $E = \mathbb{C}$ we simply write $L^p(\mu,E) = L^p(\mu)$ and $L^{1,\infty}(\mu,E) = L^{1,\infty}(\mu)$.

Fix $m > 0$ (not necessarily an integer). Define $\triangle = \{(x,x):\ x \in X\}$. A vector-valued m-dimensional Calderón-Zygmund kernel with respect to ρ is a continuous mapping $K:\ X \times X \backslash \triangle \mapsto E$ for which we have

(a) there exists a constant $C_1 > 0$ such that
$$\|K(x,y)\|_E \leq \frac{C_1}{\rho(x,y)}, \quad \forall x,y \in X \times X\backslash\triangle;$$
(b) there exist constants $0 < \epsilon \leq 1$ and $C_2, C_3 > 0$ such that
$$\|K(x,y) - K(x',y)\|_E + \|K(y,x) - K(y,x')\|_E \leq C_2 \frac{\rho(x,x')^\epsilon}{\rho(x,y)^{m+\epsilon}}$$
whenever $x,x',y \in X$ and $\rho(x,x') \leq C_3\rho(x,y)$.

Given a vector-valued m-dimensional Calderón-Zygmund kernel K, we can define (at least formally) a Calderón-Zygmund singular integral operator associated with this kernel by
$$Tf(x) = \int_X K(x,y)f(y)d\mu(x).$$

Proposition 5.1. *Let E be a Banach space. If a Calderón-Zygmund singular integral operator T is bounded from $L^q(\mu)$ into $L^q(\mu,E)$ for some fixed $1 \leq q < \infty$, then T can be extended to an operator on $L^p(\mu)$ for every $1 \leq p < \infty$ such that*
(a) *T is L^p-bounded for every $1 < p < \infty$, i.e., $\|Tf\|_{L^p(\mu,E)} \leq C_p\|f\|_{L^p(\mu)}$;*
(b) *T is of weak type $(1,1)$, i.e., $\|Tf\|_{L^{1,\infty}(\mu,E)} \leq C\|f\|_{L^1(\mu)}$ for all $f \in L^1(\mu)$;*

(c) *T is bounded from $L^\infty(\mu)$ into* $\mathrm{BMO}(X,\rho,\mu;E)$;
(d) *T is bounded from* $\mathrm{H}^1(X,\rho,\mu)$ *into* $L^1(\mu)$.

$\mathrm{H}^1(X,\rho,\mu)$ and $\mathrm{BMO}(X,\rho,\mu;E)$ can be defined in a natural way, see [11] for the details. This result must be known for experts in the field of vector-valued harmonic analysis, and the proof can be obtained by merely modifying the proof of Theorem V.3.4 in [13].

5.2. **Bergman singular integral operators.** We now turn our attention to the special case of the unit ball $\mathbb{B}_n$. Recall that

$$\varrho(z,w)=\begin{cases}\big||z|-|w|\big|+\Big|1-\dfrac{1}{|z||w|}\langle z,w\rangle\Big|, & \text{if } z,w\in\mathbb{B}_n\backslash\{0\},\\ |z|+|w|, & \text{otherwise.}\end{cases}$$

It is know that ϱ is a pseudo-metric on $\mathbb{B}_n$ and $(\mathbb{B}_n,\varrho,v_\alpha)$ is a homogeneous space for $\alpha>-1$ (e.g., Lemma 2.10 in [17]).

Let E be a Banach space. Suppose $\alpha>-1$. We are interested in vector-valued Bergman type integral operators on the unit ball $\overline{\mathbb{B}}_n$ in $\mathbb{C}^n$. More precisely, we are interested in Bergman type integral operators whose kernels with values in E satisfy the following estimates

$$\|K(z,w)\|_E\le\frac{C}{\varrho(z,w)^{n+1+\alpha}},\quad \forall(z,w)\in\mathbb{B}_n\times\mathbb{B}_n\setminus\{(\zeta,\zeta):\ \zeta\in\mathbb{B}_n\},\tag{5.1}$$

and

$$\|K(z,w)-K(z,\zeta)\|_E+\|K(w,z)-K(\zeta,z)\|_E\le\frac{C\varrho(w,\zeta)^\beta}{\varrho(z,\zeta)^{n+1+\alpha+\beta}},\tag{5.2}$$

for $z,w,\zeta\in\mathbb{B}_n$ so that $\varrho(z,\zeta)\ge\delta\varrho(w,\zeta)$, with some (fixed) $\alpha>-1,\delta>0$, and $0<\beta\le1$. That is, K is $(n+1+\alpha)$-dimensional Calderón-Zygmund kernel K with values in E on the homogeneous space $(\mathbb{B}_n,\varrho,v_\alpha)$.

Once the kernel has been defined, then a α-time Bergman singular integral operator T is defined as a Calderón-Zygmund singular integral operator with a vector-valued kernel K by

$$Tf(z)=\int_{\mathbb{B}_n}K(z,w)f(w)dv_\alpha(w),\quad z,w\in\mathbb{B}_n.\tag{5.3}$$

If T is bounded from L^p_α into $L^p(\mathbb{B}_n,v_\alpha;E)$ for any $1<p<\infty$, we call it a α-time Bergman integral operator (BIO). We denote by $\mathrm{BIO}_\alpha(E)$ all such operators. If $E=\mathbb{C}$ we write $\mathrm{BIO}_\alpha(\mathbb{C})=\mathrm{BIO}_\alpha$.

The examples that we keep in mind are the Bergman projection operator P_α from L^2_α onto $\mathcal{A}^2_\alpha$, which can be expressed as

$$P_\alpha f(z)=\int_{\mathbb{B}_n}K_\alpha(z,w)f(w)dv_\alpha(w),\quad \forall f\in L^1(\mathbb{B}_n,dv_\alpha),$$

where

$$K_\alpha(z,w)=\frac{1}{\big(1-\langle z,w\rangle\big)^{n+1+\alpha}},\quad z,w\in\mathbb{B}_n\text{ with }\alpha>-1.\tag{5.4}$$

Indeed, we have

Proposition 5.2. *(Proposition 2.13 in* [17]*)*

(i) *there exists a constant $C_1 > 0$ such that*

$$|K_\alpha(z,w)| \le \frac{C_1}{\varrho(z,w)^{n+1+\alpha}}, \quad \forall z, w \in \mathbb{B}_n.$$

(ii) *There are two constants $C_2, C_3 > 0$ such that for all $z, w, \zeta \in \mathbb{B}_n$ satisfying*

$$\varrho(z,\zeta) > C_2 \varrho(w,\zeta)$$

one has

$$|K_\alpha(z,w) - K_\alpha(z,\zeta)| \le C_3 \frac{\varrho(w,\zeta)^{\frac{1}{2}}}{\varrho(z,\zeta)^{n+1+\alpha+\frac{1}{2}}}.$$

It is well known that P_α extends to a bounded operator on L^p_α for $1 < p < \infty$ (e.g., Theorem 2.11 in [20]). Thus, we have $P_\alpha \in \mathrm{BIO}_\alpha$. This fact will be also concluded from the following result, which is clearly a special case of Proposition 5.1.

Theorem 5.1. *Let E be a Banach space and $\alpha > -1$. Suppose T is a Calderón-Zygmund singular integral operator associated with a kernel satisfying* (5.1) *and* (5.2)*. If T is bounded on $L^q(v_\alpha)$ for some fixed $1 < q < \infty$, then T is bounded from $L^p(\mathbb{B}_n, v_\alpha)$ into $L^p(\mathbb{B}_n, v_\alpha; E)$ for every $1 < p < \infty$, and is of weak type* $(1,1)$.

5.3. **Area functions as vector-valued Bergman integral operators.** Given $\gamma > 0$ and $1 < q < \infty$. Let $E = L^q(\mathbb{B}_n, \chi_{D(0,\gamma)} d\tau)$. We consider the operator

$$[T_{\mathrm{tent}} f(z)](w) = \int_{\mathbb{B}_n} \frac{f(u)dv_\alpha(u)}{(1 - \langle \varphi_z(w), u\rangle)^{n+1+\alpha}}, \quad \forall z, w \in \mathbb{B}_n, \tag{5.5}$$

with the kernel

$$K_{\mathrm{tent}}(z,u)(w) = \frac{1}{(1 - \langle \varphi_z(w), u\rangle)^{n+1+\alpha}}.$$

By the reproduce kernel formula (e.g., Theorem 2.2 in [20]) we have

$$[T_{\mathrm{tent}} f(z)](w) = f(\varphi_z(w)), \quad \forall f \in \mathcal{H}(\mathbb{B}_n),$$

and hence

$$\|T_{\mathrm{tent}} f(z)\|_E = A^{(q)}_\gamma(f)(z),$$

for any $f \in \mathcal{H}(\mathbb{B}_n)$.

Theorem 5.2. *Let $\gamma > 0, \alpha > -1, 1 < q < \infty$, and $1 < p < \infty$. Then $T_{\mathrm{tent}} \in \mathrm{BIO}_\alpha(E)$. Consequently,*

$$\|A^{(q)}_\gamma(f)\|_{L^p_\alpha} \lesssim \|f\|_{L^p_\alpha}, \quad \forall f \in \mathcal{A}^p_\alpha(\mathbb{B}_n).$$

Proof. Let $f \in L^q_\alpha(\mathbb{B}_n)$. Then

$$\begin{aligned}
\|T_{\text{tent}} f\|^q_{L^q(v_\alpha, E)} &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left| \int_{\mathbb{B}_n} \frac{f(u) dv_\alpha(u)}{(1 - \langle \varphi_z(w), u\rangle)^{n+1+\alpha}} \right|^q \chi_{D(0,\gamma)}(w) d\tau(w) dv_\alpha(z) \\
&= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |P_\alpha f(w)|^q \chi_{D(z,\gamma)}(w) d\tau(w) dv_\alpha(z) \\
&\approx \|P_\alpha f\|^q_{L^q_\alpha} \lesssim \|f\|^q_{L^q_\alpha}
\end{aligned}$$

by the L^q-boundedness of P_α for $1 < q < \infty$. This concludes that T_{tent} is bounded from L^q_α into $L^q(v_\alpha, E)$.

By Theorem 5.1, it remains to show that K_{tent} satisfies the conditions (5.1) and (5.2). It is easy to check the condition (5.1). Indeed, by Lemmas 2.1 and 2.3 and Proposition 5.2 (i) we have

$$\begin{aligned}
\|K_{\text{tent}}(z, u)\|_E &= \Big(\int_{\mathbb{B}_n} \frac{1}{|1 - \langle \varphi_z(w), u\rangle|^{2(n+1+\alpha)}} \chi_{D(0,\gamma)}(w) d\tau(w) \Big)^{\frac{1}{2}} \\
&= \Big(\int_{D(z,\gamma)} \frac{1}{|1 - \langle w, u\rangle|^{2(n+1+\alpha)}} d\tau(w) \Big)^{\frac{1}{2}} \\
&\lesssim \frac{1}{|1 - \langle z, u\rangle|^{n+1+\alpha}} \\
&\leq \frac{C}{\varrho(z, u)^{n+1+\alpha}}.
\end{aligned}$$

This concludes that K_{tent} satisfies (5.1).

To check the condition (5.2), we need the following variant of Proposition 5.2 (ii).

Lemma 5.1. *There exist two constants $C_1, C_2 > 0$ such that for all $z, u, \zeta \in \mathbb{B}_n$ satisfying*

$$\varrho(z, \zeta) > C_1 \varrho(u, \zeta)$$

one has

$$|K_\alpha(w, u) - K_\alpha(w, \zeta)| \leq C_2 \frac{\varrho(u, \zeta)^{\frac{1}{2}}}{\varrho(z, \zeta)^{n+1+\alpha+\frac{1}{2}}},$$

for all $w \in D(z, \gamma)$.

The proof can be obtained by slightly modifying the proof of Proposition 2.13 (2) in [17] with the help of Lemmas 2.2 and 2.3. We omit the details.

Now we turn out to proceed our proof. Suppose $z, u, \zeta \in \mathbb{B}_n$. Note that

$$
\begin{aligned}
&\|K_{\text{tent}}(z,u) - K_{\text{tent}}(z,\zeta)\|_E \\
&\quad =\Big(\int_{\mathbb{B}_n} \Big|\frac{1}{(1-\langle\varphi_z(w),u\rangle)^{n+1+\alpha}} - \frac{1}{(1-\langle\varphi_z(w),\zeta\rangle)^{n+1+\alpha}}\Big|^2 \chi_{D(0,\gamma)}(w)d\tau(w)\Big)^{\frac{1}{2}} \\
&\quad =\Big(\int_{\chi_{D(z,\gamma)}} \Big|\frac{1}{(1-\langle w,u\rangle)^{n+1+\alpha}} - \frac{1}{(1-\langle w,\zeta\rangle)^{n+1+\alpha}}\Big|^2 d\tau(w)\Big)^{\frac{1}{2}} \\
&\quad =\Big(\int_{\chi_{D(z,\gamma)}} \Big|K_\alpha(w,u) - K_\alpha(w,\zeta)\Big|^2 d\tau(w)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then, by Lemma 5.1 there exist two constants $C_1, C_2 > 0$ such that for all $z, u, \zeta \in \mathbb{B}_n$,

$$
\|K_{\text{tent}}(z,u) - K_{\text{tent}}(z,\zeta)\|_E \le C_2 \frac{\varrho(u,\zeta)^{\frac{1}{2}}}{\varrho(z,\zeta)^{n+1+\alpha+\frac{1}{2}}}
$$

whenever $\varrho(z,\zeta) > C_1 \varrho(u,\zeta)$.

On the other hand, since

$$
\begin{aligned}
&\|K_{\text{tent}}(u,z) - K_{\text{tent}}(\zeta,z)\|_E \\
&\quad =\Big(\int_{\mathbb{B}_n} \Big|\frac{1}{(1-\langle\varphi_u(w),z\rangle)^{n+1+\alpha}} - \frac{1}{(1-\langle\varphi_\zeta(w),z\rangle)^{n+1+\alpha}}\Big|^2 \chi_{D(0,\gamma)}(w)d\tau(w)\Big)^{\frac{1}{2}} \\
&\quad =\Big(\int_{\mathbb{B}_n} \Big|K_\alpha(\varphi_u(w),z) - K_\alpha(\varphi_\zeta(w),z)\Big|^2 \chi_{D(0,\gamma)}(w)d\tau(w)\Big)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\varrho(\varphi_u(w),\varphi_\zeta(w)) \lesssim |1-\langle\varphi_u(w),\varphi_\zeta(w)\rangle| \approx |1-\langle u,\zeta\rangle|, \quad \forall w \in D(0,\gamma),
$$

by Lemma 2.3 and the inequality

$$
\varrho(z,w) \lesssim |1-\langle z,w\rangle|
$$

(e.g., Eq.(6) in [17]), then by slightly modifying the proof of Proposition 2.13 (2) in [17] we can prove that

$$
\|K_{\text{tent}}(u,z) - K_{\text{tent}}(\zeta,z)\|_E \lesssim \frac{\varrho(u,\zeta)^{\frac{1}{2}}}{\varrho(z,\zeta)^{n+1+\alpha+\frac{1}{2}}}.
$$

The details are left to readers. This completes the proof. □

Evidently, we can define:

(i) The radial area integral operator

$$
[T_{\text{radial}} f(z)](w) = \int_{\mathbb{B}_n} K_{\text{radial}}(z,u)(w) f(u) dv_\alpha(u), \quad \forall z, w \in \mathbb{B}_n, \tag{5.6}
$$

with the Bergman kernel

$$
K_{\text{radial}}(z,u)(w) = (n+1+\alpha)\frac{(1-|\varphi_z(w)|^2)\langle\varphi_z(w),u\rangle}{(1-\langle\varphi_z(w),u\rangle)^{n+2+\alpha}}, \quad \forall z,u,w \in \mathbb{B}_n.
$$

It is easy to check that

$$[T_{\text{radial}}f(z)](w) = (1-|\varphi_z(w)|^2)\mathcal{R}f(\varphi_z(w))$$

for any $f \in \mathcal{H}(\mathbb{B}_n)$.

(ii) The complex gradient area integral operator

$$[T_{\text{grad}}f(z)](w) = \int_{\mathbb{B}_n} K_{\text{grad}}(z,u)(w)f(u)dv_\alpha(u), \quad \forall z,w \in \mathbb{B}_n, \tag{5.7}$$

with the Bergman kernel

$$K_{\text{grad}}(z,u)(w) = \frac{(n+1+\alpha)(1-|\varphi_z(w)|^2)\bar{u}}{(1-\langle\varphi_z(w),u\rangle)^{n+2+\alpha}}, \quad \forall z,u,w \in \mathbb{B}_n.$$

It is easy to check that

$$[T_{\text{grad}}f(z)](w) = (1-|\varphi_z(w)|^2)\nabla f(\varphi_z(w))$$

for any $f \in \mathcal{H}(\mathbb{B}_n)$.

(iii) The invariant gradient area integral operator

$$[T_{\text{invgrad}}f(z)](w) = \int_{\mathbb{B}_n} K_{\text{invgrad}}(z,u)(w)f(u)dv_\alpha(u), \quad \forall z,w \in \mathbb{B}_n, \tag{5.8}$$

with the Bergman kernel

$$K_{\text{invgrad}}(z,u)(w) = (n+1+\alpha)\frac{(1-|\varphi_z(\omega)|^2)^{n+1+\alpha}\overline{\varphi_{\varphi_z(\omega)}(u)}}{|1-\langle\varphi_z(\omega),u\rangle|^{2(n+1+\alpha)}}, \quad \forall z,u,w \in \mathbb{B}_n.$$

It is easy to check that

$$[T_{\text{invgrad}}f(z)](w) = \tilde{\nabla}f(\varphi_z(w))$$

for any $f \in \mathcal{H}(\mathbb{B}_n)$.

Similarly, we have

Theorem 5.3. *Let $\gamma > 0, 1 < q < \infty$, and $\alpha > -1$. Then $T_{\text{radial}}, T_{\text{grad}}$, and T_{invgrad} are all in $\text{BIO}_\alpha(E)$. Consequently, $A_\gamma^{(q)}(\mathcal{R}f), A_\gamma^{(q)}(\nabla f)$, and $A_\gamma^{(q)}(\tilde{\nabla}f)$ are all bounded on $\mathcal{A}_\alpha^p$ for every $1 < p < \infty$.*

The proof is the same as that of Theorem 5.2 and the details are omitted.

References

[1] P.Ahern and J.Bruna, Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of $\mathbb{C}^n$, *Revista Mate.Iber.* **4** (1988), 123-153.

[2] P.Ahern and W.Cohn, Besov spaces, Sobolev spaces, and Cauchy integrals, *Michigan Math.J.* **39** (1972), 239-261.

[3] A.Aleksandrov, Function theory in the unit ball, in: *Several Complex Variables II* (G.M. Khenkin and A.G. Vitushkin, editors), Springer-Verlag, Berlin, 1994.

[4] N.Arcozzi, R.Rochberg, and E.Sawyer, Carleson measures and interpolating sequences for Besov spaces on complex balls, *Memoirs Amer. Math. Soc.* **859** (2006), no. 859, vi+163 pp.

[5] N.Arcozzi, R.Rochberg, and E.Sawyer, Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls, *Adv. Math.* **218** (2008), 1107-1180.

[6] F.Beatrous and J.Burbea, Holomorphic Sobolev spaces on the ball, *Dissertationes Mathematicae* **276**, Warszawa, 1989.

[7] Z.Chen and W.Ouyang, Maximal and aera integral characterizations of Bergman spaces in the unit ball of $\mathbb{C}^n$, arXiv: 1005.2936.

[8] Z.Chen and W.Ouyang, Real-variable characterizations of Bergman spaces in the unit ball of $\mathbb{C}^n$, arXiv: 1103.6122.

[9] R.R.Coifman, Y.Meyer and E.M.Stein, Some new function spaces and their applications on harmonic analysis, *J. Funct. Anal.,* **62** (1985), 304-335.

[10] R.Coifman and R.Rochberg, Representation theorems for holomorphic and harmonic functions in L^p, *Asterisque* **77** (1980), 11-66.

[11] R.Coifman and G.Weiss, Extension of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569-643.

[12] P.Duren and A.Schuster, *Bergman spaces*, Mathematical Surveys and Monographs **100**, American Mathematical Society, Providence, RI, 2004.

[13] J.Garcia-Cuerva and J.L.Rubio de Francia, Weighted norm inequalities and related topics, North Holland, Amsterdam, 1985.

[14] J.Garnett and R.H.Latter, The atomic decomposition for Hardy space in several complex variables, *Duke J.Math.* **45** (1978), 815-845.

[15] F.Nazarov, S.Treil, and A.Volberg, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, *Internat.Math.Res.Notices* **9** (1998), 463-487.

[16] E.Stein, *Harmonic analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals,* Princeton University Press, Princeton, NJ, 1993.

[17] E.Tchoundja, Carleson measures for the generalized Bergman spaces via a $T(1)$-type theorem, *Ark. Mat.* **46** (2008), 377-406.

[18] A.Volberg and B.D.Wick, Bergman-type singular integral operators and the characterization of Carleson measures for Besov-Sobolev spaces on the complex ball, arXiv: 0910.1142, *Amer.J.Math.,* to appear.

[19] R.Zhao and K.Zhu, Theory of Bergman spaces in the unit ball of $\mathbb{C}_n$, *Memoires de la Soc.Math.France* **115** (2008), pp.103.

[20] K.Zhu, *Spaces of Holomorphic Functions in the Unit Ball,* Springer-Verlag, New York, 2005.

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, 30 West District, Xiao-Hong-Shan, Wuhan 430071, China

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, 30 West District, Xiao-Hong-Shan, Wuhan 430071, China and Graduate School, Chinese Academy of Sciences, Wuhan 430071, China